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SUPPORT AND CALIBRATION FUNCTIONS OF REACHABILITY REGIONS OF LINEAR CONTROLLED SYSTEMS*

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Linear non-autonomous systems with convex simultaneous contraints on the initial state and the control are considered. Because of its convexity, the constraint set can be described in terms of a support or a calibration function. The reachability region (RR) is also convex. Formulas are derived for the support and calibration functions of the reachability region, using the corresponding functions of the constraint set. Methods of RR construction and estimation are proposed on the basis of these formulas.

1. *Basic relationships.*

Consider the system

$$x^{\cdot} = A(t)x + B(t)u, \quad t \in T = [t_0, t_1] \tag{1.1}$$

Here x is an n-dimensional phase vector, u is the m-dimensional control vector, which is a summable function of time t, and $A(t)$ and $B(t)$ are appropriately dimensioned continuous matrices. In the space of initial states and controls $\Lambda = \{\lambda = (x_0, u(t))\} = R^n \times L$ (L is the set of m-dimensional vector-functions summable on T), the feasible initial states and controls belong to the constraint set Ω. For a given control $u(t)$ and a given initial condition x_0, the solution of system (1.1) is expressed by the Cauchy formula (here and henceforth integration is from t_0 to t)

$$x(t) = H(t, t_0, x_0, u) = \Phi(t, t_0)x_0 + \int \Psi(t, \tau)u(\tau)\,d\tau \tag{1.2}$$

which defines the linear operator $H(t, t_0, \circ, \circ)\colon \Lambda \to R^n$. We introduce the operator H_e acting from the quotient-space Λ/N to R^n by the formula $H = H_eE$, where $N = \ker H$, Λ/N is the quotient-space of Λ by the subspace N and E is the natural mapping that associates to the element $\lambda \in \Lambda$ its equivalence class $E\lambda = [\lambda]$.

The reachability region (RR) $Q(t_0, t) = \{x \in R^n\colon x = H(t, t_0, x_0, u),\ (x_0, u(t)) \in \Omega\}$ is the image of the constraint set Ω under the mapping $H: Q = H(t, t_0, \circ, \circ)\Omega$. Let us investigate this region for the case when Ω is a convex set.

2. *The boundary point condition.*

In the space Λ consider the linear functional

$$\langle (y_0, v(\tau)), (x_0, u(\tau))\rangle = \langle y_0, x_0\rangle_n + \int \langle v(\tau), u(\tau)_m\rangle\,d\tau \tag{2.1}$$

$$\langle x, y\rangle_n = \sum_{i=1}^{n} x^i y^i, \quad x^T = (x^1, \ldots, x^n), \quad y_0 \in R^n$$

where the set of functions $v(\tau)$ is defined by the constraint set Ω. Thus, for

$$\Omega = \left\{(x_0, u(\tau))\colon x_0 = 0, \ \int |u(\tau)|\,d\tau \leqslant A, \ u \in R^1\right\}$$

$v(\tau)$ are the essentially bounded functions.

From (1.2), (2.1) it follows that the equivalence class of the mapping H in the space Λ of initial states and controls is a plane manifold that belongs to the intersection of the family of hyperplanes

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$$x^i = \langle \gamma_i, \lambda \rangle, \ i = 1, 2, \ldots, n \tag{2.2}$$
$$\gamma_i^T = (\varphi_{i*}^T(t), \psi_{i*}^T(t, \tau)), \quad \Phi^T(t, t_0) = (\varphi_{1*}(t), \ldots, \varphi_{n*}(t))$$
$$\Psi^T(t, \tau) = (\psi_{1*}(t, \tau), \ldots, \psi_{n*}(t, \tau))$$

Theorem 1. The equivalence class of mapping H defines a boundary point of the RR if it completely belongs to one of the supporting hyperplanes of the constraint set Ω in the space Λ.

In order to prove the theorem, note that if the equivalence class $[\lambda]$ contains an interior point $\lambda_b \in \Omega$, then taking a neighbourhood of this point in Ω and acting on it by the operator H, we obtain that the corresponding point $x = H\lambda_b$ is an interior point of the RR. Hence it follows that the equivalence classes defining the RR boundary points do not contain interior points of the constant set, i.e., they completely belong to some supporting hyperplanes of the set Ω.

Using the description (2.2) of the equivalence classes, we obtain the boundary point condition from Theorem 1: a RR boundary point is determined by an equivalence class for which the normal γ to the supporting hyperplane of the set Ω is a linear combination of the vectors γ_i introduced in (2.2),

$$\gamma = \alpha_1\gamma_1 + \ldots + \alpha_n\gamma_n, \quad \alpha_i = \text{const}, \ i = 1, \ldots, n \tag{2.3}$$

Formulas (1.2), (2.3) and the formula for the boundary of the constraint set provide a parametric description of the RR boundary.

As an example, let us construct the RR of system (1.1) with the constraint set

$$\Omega_0 = \left\{(x_0, u(\tau)) : \|x_0\|_n^2 + \int \|u(\tau)\|_m^2 d\tau \leqslant A^2\right\} \tag{2.4}$$

The boundary of the set Ω_0 is defined by the equation $F(\lambda) = \langle (x_0, u), (x_0, u)\rangle - A^2 = 0$, and therefore the normal vector is $\gamma = \operatorname{grad}_\lambda F = (x_0, u)$. The boundary point condition (2.3) is written in the form

$$\chi\alpha = \begin{Vmatrix} x_0 \\ u \end{Vmatrix}, \quad \chi = \begin{Vmatrix} \Phi^T(t, t_0) \\ \Psi^T(t, \tau) \end{Vmatrix}, \quad \alpha^T = (\alpha_1, \ldots, \alpha_n)$$

whence we obtain

$$\alpha = \Phi^{T-1}(t, t_0)x_0, \quad u = \Psi(t, \tau)\Phi^{T-1}(t, t_0)x_0$$

Substituting into (1.2), we obtain

$$x_0 = \Phi^T(t, t_0)Y^{-1}(t, t_0)x, \ u = \Psi^T(t, \tau)Y^{-1}(t, t_0)x$$
$$Y(t, t_0) = \Phi(t, t_0)\Phi^T(t, t_0) + \int \Psi(t, \tau)\Psi^T(t, \tau)\,d\tau$$

Substituting these values in the equations of the boundary of the set Ω_0, we find that the RR is an ellipsoid with the boundary

$$(Y^{-1}(t, t_0)x, x)_n = A^2 \tag{2.5}$$

For the case when the initial set consisits of a single point x_0, this is a well-known result /1/.

3. *The support function of the RR.*

The convexity of the constraint set and the existence of the scalar product (2.1) in the space Λ make it possible to compute the support function $q_\Omega(\gamma)$ of the set Ω. Let us express the support function $q_Q(\beta)$ of the RR in terms of $q_\Omega(\gamma)$. We have

$$q_Q(\beta) = \sup_{x \in Q}(\beta, x)_n = \sup_{\lambda \in \Omega}\left[(\Phi(t, t_0)x_0, \beta)_n + \int(\Psi(t, \tau)u(\tau), \beta)_n d\tau\right] =$$
$$\sup_{\lambda \in \Omega}\langle \chi\beta, \lambda\rangle = q_\Omega(\chi\beta)$$

Hence we obtain for the RR support function

$$q_Q(\beta) = q_\Omega(\chi\beta) \tag{3.1}$$

We use (3.1) to compute the support function of the RR of the system (1.1) for three types of constraints: Ω_0 (formula (2.4)), Ω_1, and Ω_2, where

$$\Omega_1 = \{(x_0, u(\tau)) : \max(|x_0^i|, \sup_{\tau\in T} |u^j(\tau)|, i=1,\ldots,n;\ j=1,\ldots,m) \leqslant A\}$$
$$\Omega_2 = \left\{(x_0, u(\tau)) : \sum_{i=1}^{n} |x_0^i| + \sum_{j=1}^{m} \sup_{\tau\in T} |u^j(\tau)| \leqslant A\right\} \tag{3.2}$$

Let us show the detailed calculations for Ω_1. From the definition of the support function, we obtain from (3.1), (3.2)

$$q_{Q_1}(\beta) = \sup_{\lambda\in\Omega_1} \left\{(\Phi^T(t,t_0)\beta, x_0)_n + \int (\Psi^T(t,\tau)\beta, u(\tau))_m d\tau\right\} \leqslant$$
$$\sup_{\lambda\in\Omega_1} \left\{\sum_{i=1}^{n} |(\varphi_i(t,t_0),\beta)_n||x_0^i| + \sum_{j=1}^{m} \sup_{\tau\in T} |u^j(\tau)| \times\right.$$
$$\left.\int |(\psi_j(t,\tau),\beta)_n| d\tau\right\} \leqslant A\left\{\sum_{i=1}^{n} |(\varphi_i(t,t_0),\beta)_n| + \sum_{j=1}^{m} \int |(\psi_j(t,\tau),\beta)_n| d\tau\right\} \tag{3.3}$$
$$\Phi(t,t_0) = (\varphi_1(t,t_0),\ldots,\varphi_n(t,t_0)),\ \Psi(t,\tau) = (\psi_1(t,\tau),\ldots,\psi_m(t,\tau))$$

The equality in (3.3) is attained when

$$x_{01}^i = A\,\mathrm{sign}\,(\varphi_i(t,t_0),\beta)_n,\quad u_1^j(\tau) = A\,\mathrm{sign}\,(\psi_j(t,\tau),\beta)_n$$

Therefore, for the support function $q_{Q_1}(\beta)$ we obtain the expression on the right-hand side of the chain of inequalities (3.3).

We similarly obtain expressions for the support functions and the corresponding values of the initial state and control for the sets Ω_0 and Ω_2:

$$q_{Q_0}(\beta) = A\left\{\sum_{i=1}^{n} (\varphi_i(t,t_0),\beta)_n^2 + \sum_{j=1}^{m} \int (\psi_j(t,\tau),\beta)_n^2 d\tau\right\}^{1/2}$$
$$x_{00}^i = A^2(\varphi_i(t,t_0),\beta)_n / q_{Q_0}(\beta),\ u_0^j(\tau) = A^2(\psi_j(t,\tau),\beta)_n / q_{Q_0}(\beta)$$
$$q_{Q_2}(\beta) = A\max\left\{|(\varphi_i(t,t_0),\beta)_n|, \int |(\psi_j(t,\tau),\beta)_n| d\tau,\quad i=1,\ldots,n;\ j=1,\ldots,m\right\}$$
$$x_{02}^i = \begin{cases} Ak^{-1}\,\mathrm{sign}\,(\varphi_i(t,t_0),\beta)_n, & \xi_i = A \\ 0, & \xi_i \neq A\end{cases}$$
$$u_2^j(\tau) = \begin{cases} Ak^{-1}\,\mathrm{sign}\,(\psi_j(t,\tau),\beta)_n, & \eta_j = A \\ 0, & \eta_j \neq A\end{cases}$$
$$\xi_i = |(\varphi_i(t,t_0),\beta)_n|,\quad \eta_j = \sup_{\tau\in T} |(\psi_j(t,\tau),\beta)_n|$$

k is the number of variables ξ_i, η_j $(i=1,\ldots,n;\ j=1,\ldots,m)$ that are equal to the maximum value A.

RR support functions for the case when the constraints are imposed only on the control were obtained in /1, 2/.

4. *Calibration function of the RR.*

The constraint set is assumed to be absorbing in its linear hull $L(\Omega)$ (if this is not so, then it is first made absorbing by a translation). Then the calibration function of the set Ω, $p_\Omega(\lambda) = \inf\{r > 0 : \lambda \in r\Omega\}$, is a convex functional. In $\Lambda\,|\,N$ define the functional

$$P(\xi) = \inf_{\lambda\in\xi} p_\Omega(\lambda) \tag{4.1}$$

The following theorem was proved in /3/ for the calibration function of the reachability region $p_Q(x)$.

Theorem 2. The functional PH_e^{-1} is a calibration function of the RR.

In order to apply this theorem, we need the description of the equivalence class $H_e^{-1}x$. By direct substitution we verify that $H_e^{-1}x$ consists of the elements $\lambda \in \Lambda$ defined by the multivalued mapping

$$F(x) = \left\| \begin{matrix} C_1 D^{-1} x \\ C_2(\tau) D^{-1} x \end{matrix} \right\|,\quad D = \Phi(t,t_0) C_1 + \int \Psi(t,\tau) C_2(\tau) d\tau \tag{4.2}$$

where $C_1, C_2(\tau)$ are matrices with arbitrary elements that produce a non-singular matrix D.

5. *Application in control problems.* The formulas obtained for the calibration and support functions may be used to construct and estimate the RR and also to solve other problems of control theory. Let us suggest some applications.

From Theorem 2 we have

$$\operatorname{int} Q = \{x \in R^n: PH_\varepsilon^{-1}x < 1\},\ \partial Q = \{x \in R^n: PH_\varepsilon^{-1}x = 1\} \tag{5.1}$$

The main difficulty in applying these formulas is the evaluation of the functional P. If we use only approximate values, then (5.1), (4.2) give inner estimates of the RR. Let $\mu(x)$ be some single-valued section of the mapping $F(x)$. Then an inner estimate is given by the formula

$$Q_b = \{x \in R^n: p_\Omega(\mu(x)) < 1\} \tag{5.2}$$

Choosing an appropriate $\mu(x)$, we can ensure any prespecified accuracy of the estimate (5.2).

Inner and outer estimate of the RR can be obtained if the calibration function of the constraint set Ω is comparable with the calibration function of another set Ω': $p_\Omega(\lambda) \leqslant p_{\Omega'}(\lambda)$ $(p_\Omega(\lambda) \geqslant p_{\Omega'}(\lambda))$. Then from Theorem 2 and formulas (5.1), (4.2) we obtain the inclusion $\bar{Q} \supset Q'(\bar{Q}' \supset Q)$, where Q' is the RR of system (1.1) with the constraint set Ω'.

Thus, for system (1.1) with the constraint set Ω_0, the RR is an ellipsoid with the boundary (2.5). Therefore, for systems whose constraints are comparable with (2.4) we can obtain ellipsoidal estimates of the RR. In particular, we have the inequalities $p_{\Omega_1} > p_{\Omega_0}$, $p_{\Omega_0} > p_{\Omega_2}$; our results for system (1.1) with the constraint set Ω_1 produce the outer ellipsoidal estimate (2.5), and the results with the constraint set Ω_2 produce an inner estimate.

There are advantages to applying the support function for RR analysis. The boundary of the RR can be defined as the envelope family of support hyperplanes $(x, \beta)_n = q_Q(\beta)$, and therefore at its points of differentiability the support function is defined by the equations $(x, \beta)_n = q_Q(\beta)$, $x = \partial q_Q/\partial\beta$. It is fairly easy to obtain an estimate by polyhedra $(x, \beta_i)_n \leqslant q_Q(\beta_i)$, fixing the normals β_i to the faces, whose choice may be determined by some additional conditions specified in the problem. Thus, in observation problems, a useful estimate is by rectangles $|x^i| \leqslant q_Q(e_i)$ $i = 1, 2, \ldots n$, where e_i is the unit vector along the axis Ox^i. By increasing the number of faces, we can achieve the required estimation accuracy.

Formula (3.1) gives a solution of the problem of visiting the hyperplane $(\alpha, x)_n = d$, defined by the condition $q_Q(\alpha) = d$.

If we have $q_{Q_1}(\beta) \geqslant q_{Q_2}(\beta)\ \forall\beta$, then clearly $Q_1 \supseteq Q_2$. This can be used to obtain a comparative estimate of the RR. In particular, from the results of Sect.3 we have $q_{Q_1} > q_{Q_0} > q_{Q_2}$, and therefore $Q_1 \supset Q_0 \supset Q_2$, which agrees with the result obtained using calibration functions. Moreover, identifying in the formula for the support function the terms that depend on the initial state (characterized by the functions $\varphi_i(t_1, t_0)$) and on the control (characterized by the functions $\psi_i(t, \tau)$), we can estimate their contribution to the evolution of the RR, which is also important for the development of real control systems and control laws.

REFERENCES

1. BUBLIK B.N. and KIRICHENKO N.F., Fundamentals of Control Theory, Vishcha Shkola, Kiev, 1975.
2. FORMAL'SKII A.M., Controllability and Stability of Systems with Scarce Resources, Nauka, Moscow, 1974.
3. GUBIN S.V. and KOVALEV A.M., The calibration function of the reachability region of a linear control system and its application in estimation problems, Dokl. Akad. Nauk UkrSSR, ser. A, 10, 1987.

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